UDC 51

Alymkulov K., Kozhobekov K.G. A new approach to constructing the asymptotic of the solution of the Bessel equation for large values of the complex argument

Keldibay Alymkulov,
Doctor of physical and mathematical sciences, профессор, Director of the Institute of fundamental and applied researches at Osh state university, Correspondent member of the National academy sciences of Kyrgyzstan
Kozhobekov Kudaiberdi Gaparalievich,
D.ph., senior researcher

Abctract. *The asymptotic of the solution of the Bessel equation for large values of the complex argument is obtained directly from its differential equation.*
Keywords: *Bessel equation, method of indeterminate coefficients, principle of contracting operators, reduction to an integral equation, asymptotic of a solution.*

1. Introduction

Next Bessel equations are considered

$$\frac{d^2Y(z)}{dz^2}+\frac{1}{z}\frac{dY(z)}{dz}+\left(1-\frac{\nu^2}{z^2}\right)Y(z)=0 \tag{1}$$

which appears in many areas of science and technology, where, $z\in D=\{z: |Argz|<\frac{\pi}{2}-\delta, \delta>0\}$, $C\ni\nu$ is the order of Bessel functions , Y(z) is unknown function of the complex variable $z=x+iy, i=\sqrt{-1}$.

In [1], the asymptotic solution of equation (1) for large values of $z=x\in R$ was obtained by reducing it to the Riccati equation, and it was noted there that this asymptotic can be obtained directly from equation (1) without reducing to the Riccati equation.

In [2] the asymptotic solution of equation (1) for large values of $z=x\in R$ was obtained for large values of the argument, directly from it.

Here, the previously obtained asymptotics of the solution of the Bessel equation for a real argument [2] is generalized to the case of a complex argument.

Usually, the asymptotic of the solution of equation (1) for large values of the argument z, is obtained from the integral representation of its solution [2-8].

2. Direct very simple approach to obtaining the asymptotic of the solution from the Bessel equation

Substitution

$$Y(z)=\frac{1}{\sqrt{z}}u(z) \tag{2}$$

where u (z) is a new unknown function, equation (1) is reduced to the form

$$u^{"}(z)+\left(1+\frac{\alpha}{z^2}\right)u(z)=0, \tag{3}$$

here

$$\alpha=\frac{1}{4}(1-4v^2). \tag{4}$$

We obtain the asymptotic of the solution of problem (2) in domain $z\in D=\{z:|Argz|<\frac{\pi}{2}-\delta,\delta>0\}$.

The solution of the equation (3) we seek in next form

$$\mathrm{u}(z)=\cos \mathrm{z}\, X(z)+\mathrm{sinz}\,\mathrm{Y(z)}, \tag{5}$$

here X(z) and Y(z) new independents functions.

After substitution of (3) to (3) we have got for X(z) and Y(z) next equations

$$\frac{d^2X(z)}{dz^2}-X(z)+2\frac{dY(z)}{dz}+(1+\alpha z^{-2})X(z)=0, \tag{6.1}$$

$$\frac{d^2Y(z)}{dz^2}-Y(z)-2\frac{dX(z)}{dz}(1+\alpha z^{-2})Y(z)=0. \tag{6.2}$$

We will impose the following conditions on the functions X(z) and Y(z)

$X(z)\to 1, z\to\infty; Y(z)\to 0, z\to\infty.$

These functions we will seek in next forms

$$\mathrm{X}(z)=1+A_2z^{-2}+\ A_4z^{-4}+A_6z^{-6}+\cdots+\ A_{2m}z^{-2\mathrm{m}}, \tag{7.1}$$

$$\mathrm{Y}(z)=B_1z^{-1}+B_3z^{-3}+B_5z^{-5}+B_7z^{-7}+\cdots+B_{2m+1}z^{-(2\mathrm{m}+1)}. \tag{7.2}$$

A_k $(k=2,4,\dots), B_l$ (l=1,3,...) are indefinites numbers coefficients for the present.

$$\frac{dX(z)}{dz}=-2A_2z^{-3}-4\,A_4z^{-5}-6A_6z^{-7}-8A_8z^{-9}-2\mathrm{m}A_{2m}z^{-2\mathrm{m}-1}-\cdots,$$

$$\frac{d^2X(z)}{dz^2}=2\cdot 3A_2z^{-3}+4\ \cdot 5A_4z^{-6}+6\ \cdot 7A_6z^{-8}+8\cdot 9A_8z^{-10}+2\mathrm{m}(2m+1)A_{2m}z^{-2\mathrm{m}-2}+\cdots$$

$$\frac{dY(z)}{dz}=-B_1z^{-2}-3\,B_3z^{-4}-5B_5z^{-6}-7B_7z^{-8}-9B_9z^{-10}-\cdots-(2\mathrm{m}+1)B_{2m+1}z^{-2\mathrm{m}-2}\dots,$$

$$\frac{d^2Y(z)}{dz^2}=2B_1z^{-3}+3\cdot 4B_3z^{-5}+5\ \cdot 6B_5z^{-7}+7\cdot 8B_7z^{-9}+\cdots+(2\mathrm{m}+1)(2m+2)B_{2m-1}z^{-2\mathrm{m}-3}+\cdots$$

Substituting one and two times differentiated series (6) into (5) and equating the coefficients with the same degree z, for indefinite coefficients A_k $(k=2,4,\dots), B_l$ (l=1,3,...) we obtain next expressions

$$-2B_1\ +\alpha=0, \tag{8.1}$$

$$(2+\alpha)B_1\ +2\cdot 2A_2\ =0, \tag{8.2}$$

$$(\alpha+2\cdot 3)A_2\ -2\cdot 3B_3\ =0, \tag{8.3}$$

$$(\alpha+3\cdot 4)B_3\ +2\cdot 4A_4\ =0, \tag{8.4}$$

$$(\alpha+4\cdot 5)A_4\ -2\cdot 5B_5\ =0, \tag{8.5}$$

$$(\alpha+5\cdot 6)B_5\ +2\cdot 6A_6\ =0, \tag{8.6}$$

$$(\alpha+6\cdot 7)A_6\ -2\cdot 7B_7\ =0, \tag{8.7}$$

$$(\alpha+7\cdot 8)B_7\ +2\cdot 8A_8\ =0 \qquad \textbf{(8.8)}$$

…………………………………………………………………………

$(\alpha+(2m+1)(2m+2))B_{2m+1}\ +2\cdot(2m+2)A_{2m+2}\ =0,$ **(8.2m+2)**

$(\alpha+(2m+2)(2m+3))A_{2m+2}\ -2\cdot(2m+3)B_{2m+3}\ =0,$ **(8.2m+3)**

From this, we successively determine the unknown constants $B_{1\,,}\,A_2\,,B_{3\,,}\,A_4$ ...

$B_1=\frac{\alpha}{2}=(4)=-\frac{(4\nu^2-1)}{8},$

$A_2=-\frac{(\alpha+2)}{2\cdot 2}B_1=-\frac{(4\nu^2-1)(4\nu^2-3^2)}{2\cdot 8^2},$

$B_3=\frac{(\propto+2\cdot 3)}{3!}A_2=-\frac{\propto(\alpha+2)(\propto+2\cdot 3)}{2\cdot 2}=-\frac{(4\nu^2-1)(4\nu^2-3^2)(4\nu^2-5^2)}{3!\cdot 8^3}$

Similarly we will find

$$B_{2k+1}=(-1)^{k+1}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots(4\nu^2-(4k+1)^2))}{(2k+1)!\cdot 8^{(2k+1)}}$$

$$A_{2k+2}=(-1)^{k+1}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots(4\nu^2-(4k+3)^2))}{(2k+2)!\cdot 8^{(2k+2)}}$$

Theorem. The series (7) is an asymptotic series, i.e. if we consider a truncated series

$$X(z)=\sum_0^m A_{2k}\frac{1}{z^{2k}}+R^{(1)}{}_{2m+2}(z) \qquad \textbf{(9.1)}$$

$$Y(z)=\sum_0^m B_{2k-1}\frac{1}{z^{2k-1}}\ +R^{(2)}{}_{2m+1}(z) \qquad \textbf{(9.2)}$$

here

$$\left|R^{(1)}{}_{2m+2}(z)\ \right|\le l|z|^{-2m-2},\qquad \left|R^{(2)}{}_{2m+1}(z)\right|\ \le l|z|^{-2m-}\,,l=\mathrm{cons}\,t,\ \ z\in D,\quad z\to\infty\,.$$

Proof. For simplicity, we prove this theorem, for *m*=0.

Let

$$X(z)=\ 1+R^1(z\,), \qquad \textbf{(10.1)}$$

$$Y(z)=B_1z^{-1}\ +R^2(z\,), \qquad \textbf{(10.2)}$$

here

$$|R^1(z\,)\mid\le l|z|^{-2},\ \left||R^2(z\,)\mid\le l|z|^{-3}\right|\ ,l=\mathrm{cons}\,t,\ \ z\in D,\quad z\to\infty\,.$$

Substitution (10) to (6) we have got next equations

$$\frac{d^2R^1(z\,)}{dz^2}+2\frac{d}{dz}R^2(z\,)+\alpha z^{-2}R^1(z\,)=0$$

$$\frac{d^2R^2(z\,)}{dz^2}-2\frac{dR^1(z\,)}{dz}+\alpha z^{-2}R^2(z\,)-\gamma z^{-3}=0,$$

here $=-2B_1-4B_1^2$.

Now we introduce new function

$\mathrm{V}(z)=R^1(z\,)+iR^2(z\,);\ V(z)=O(|z|^{-2}),z\to\infty$ **(11)**

then the previous equation can be written as

$$\frac{d^2V(z)}{dz^2}-2\frac{dV(z)}{dz}+\alpha z^{-2}V(z)=\gamma z^{-3}. \qquad \textbf{(12)}$$

In equation (12) we make the following transformation

$$V(z)=e^zW(z) \qquad \textbf{(13)}$$

here $W(z)$ is new unknown function. Then we have next equation for $W(z)$

$$\frac{d^2W(z)}{dz^2} - W(z) + \alpha z^{-2}W(z) = \gamma e^{-z}z^{-3}. \quad (14)$$

The solution of this equation, which tends to zero when $z \to \infty$ will equivalent the solution of the following integral equation

$$W(z) = \int_{\infty}^{z} \sinh(z-s)\,[\gamma e^{-s}s^{-3} + \alpha s^{-2}W(s)]ds\,, \quad (15)$$

In this equation again make next transformation

$$W(z) = e^{-z}M(z).$$

Then

$$M(z) = g(z) + \alpha \int_{\infty}^{z} e^{z-s}s^{-3}\sinh(z-s)\,M(s)]ds \coloneqq T[z] \quad (16)$$

here $g(z) = \gamma \int_{\infty}^{z} e^{z-s}s^{-3}\sinh(z-s)\,ds$.

The path of integration we take the ray going from the point $z = r\,e^{i\varphi}$ to point $\infty\,e^{i\varphi}$.

Obviously, there is an estimate

$$|g(z)| \le l|z|^{-2}, z \to \infty, z \in D.$$

Denote by S the set of functions satisfying the condition

$$|M(z)| \le 2l|z|^{-2}, z \to \infty, z \in D.$$

Obviously, the operator T maps the set S to itself. Let us prove that the operator T is contractive in S. We have from the equation (16)

$$|T[M_1| - |T[M_1| \le \|M_1(z) - M_2(z)\||\alpha||z|^{-2}.$$

Let $2|\alpha||z|^{-2} \le 1$. Then the operator is the contraction operator. Theorem is proven.

Conclusion

Here, the asymptotic behavior of the solution of the Bessel equation for large values of the independent variable is obtained directly from the differential equation itself in the right complexing plane , and the asymptotic character of the solution obtained is proved.

When z and ν is real, taking into account expression (2), formula (5) coincides with previously obtained formulas with the accuracy of a constant factor [2-5].

References

1. Alymkulov K., Кожобеков К.Г (2019) . Asymptotics of the solution of the Bessel equation for large values of the argument. Mejdunarjdnyi studencheskiy nauchnyi vestnik (Russian), **1**:1-7.
2. Alymkulov K.,Kochobekov K. Asymptotics of the solution of Bessel equation at large values of the argument . Norwegian Journal of development of the International Science, № 27/ 2019, Vol.2 , 58-62.
3. Lebedev N.N. (1972) Special functions and their applications. Dover pub.
4. NIST. Handbook of mathematical functions, Chapter 10, ed. Olver F.W.J., Cambridge univ. Press, 2010
5. Olver F. (1974) Asymptotic and special functions, Academic Press, New York.
6. Temme Nico M . (2015). Asymptotic methods for integrals, World Scientific